Online Appendix
"Attributes: Selective Learning and Influence"

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Contents

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E Supplementary material for section 2

E.1 Two remarks on sample-path continuity of Gaussian processes

Remark E.1 (Independent attributes)**.** *Let $\tilde{\sigma}(a,a)=1$ and $\tilde{\sigma}(a,a')=0$ for any two distinct $a,a'\in\mathcal{A}$. This is ruled out by assumption 1. In fact, a zero-mean GP with such $\tilde{\sigma}$ has discontinuous sample paths.*

Remark E.2 (Stationary GPs that are sample-path continuous)**.** *Let $\sigma(a,a')=:g(|a-a'|)$ be the distance-based covariance of a stationary Gaussian process. To check that the process is sample-path continuous, one only needs to check that:*

(i) g is continuous at zero, i.e. $\lim_{\tau\to 0} g(\tau)=g(0)$;

(ii) there exist $\beta,K>0$ such that for any feasible distance τ such that $0\leq\tau\leq\max_{a,a'\in\mathcal{A}}|a-a'|$, $g(0)-g(\tau)\leq K\tau^{d+\beta}$.

Condition (i) guarantees that g is continuous and the process is mean-square continuous everywhere. Condition (ii) follows from proposition A.1.

E.2 Unit sum of attribute weights

In the following result, let $d^*(\nu_P(\mathbf{a},f(\mathbf{a})))\in\Delta(\{0,1\})$ denote the principal's best response to $(\mathbf{a},f(\mathbf{a}))$, where $d^*(\cdot)=1$ denotes adoption.

Lemma E.1. *Fix $k\in\mathbb{N}$, $\sigma:\mathcal{A}\times\mathcal{A}\to\mathbb{R}$, and $(\omega_A,\omega_P,r_A,r_P)$. If $(\mathbf{a}^*,d^*)$ is an equilibrium for $(\omega_A,\omega_P,r_A,r_P)$, then it is also an equilibrium for $(\tilde{\omega}_A,\tilde{\omega}_P,\tilde{r}_A,\tilde{r}_P)$ defined as*

$$\tilde{\omega}_i(a):=\frac{\omega_i(a)}{\Omega_i},\quad \tilde{r}_i:=\frac{r_i}{\Omega_i}\quad\forall i=A,P,$$

where

$$\Omega_i:=\int_{\mathcal{A}}\omega_i(a)\,\mathrm{d}a.$$

Proof. By the Lebesgue-integrability of ω_i, $|\Omega_i|<\infty$ for each i. Therefore, $\tilde{\omega}_i$ and $\tilde{r}_i$ are well-defined. It follows that $\tilde{\nu}_0^i=\nu_0^i/\Omega_i$ and $\tilde{\tau}_j^i(\mathbf{a})=\tau_j^i(\mathbf{a})/\Omega_i$ for any $i=A,P$, any $\mathbf{a}=(a_1,\ldots,a_n)\in\mathcal{A}_k$, and any $j\leq n$. This implies that for any $\mathbf{a}\in\mathcal{A}_k$, $\tilde{\psi}_i(\mathbf{a})=\psi_i(\mathbf{a})/\Omega_i$ and $\tilde{\rho}(\mathbf{a})=\rho(\mathbf{a})$. From lemma 4.1, for any given $\mathbf{a}\in\mathcal{A}_k$ the agent's payoff is

$$V_A(\mathbf{a})=r_A+(\nu_0^A-r_A)\Phi\left(\frac{\nu_0^P-r_P}{\alpha_1(\mathbf{a})}\right)+\alpha_2(\mathbf{a})\phi\left(\frac{\nu_0^P-r_P}{\alpha_1(\mathbf{a})}\right).$$

Hence $\tilde{V}_A(\mathbf{a})=V_A(\mathbf{a})/\Omega_A$. If $\mathbf{a}^*$ maximizes $V_A(\mathbf{a})$, then it also maximizes $\tilde{V}_A(\mathbf{a})$. Moreover, for any $(\mathbf{a},f(\mathbf{a}))$ if $\nu_P(\mathbf{a},f(\mathbf{a}))-r_P\geq 0$ then $\tilde{\nu}_P(\mathbf{a},f(\mathbf{a}))-\tilde{r}_P\geq 0$ as well. a □

E.3 Observational noise

Fix a sample $\mathbf{a} \in \mathcal{A}_k$ and noisy observations $y(\mathbf{a}) = f(\mathbf{a}) + \epsilon(\mathbf{a})$ where $\epsilon(a) \sim \mathcal{N}(\mu^0(a), \eta^2(a))$ for observational bias $\mu^0(\cdot)$ and noise variance $\eta^2(\cdot)$. Figure 1 illustrates interpolation with observational noise for the Brownian motion.

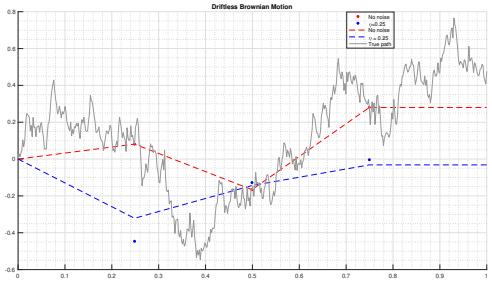


Figure 1: Extrapolation across a standard Brownian path with $\eta = 0$ (red) and $\eta = 0.25$ (blue). Sample $\mathbf{a} = (1/4, 1/2, 3/4)$. Also, $\mu^0(a) = 0$ and $\mu(a) = 0$ for all $a \in [0, 1]$.

Corollary E.2. *The set of single-player samples does not depend on observational bias μ^0. Moreover it is the same for both simultaneous and sequential sampling.*

Proof. Fix a sample $\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{A}_k$. The observations are distributed according to

$$\begin{pmatrix} y(a_1) \\ \vdots \\ y(a_k) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu(a_1) + \mu^0(a_1) \\ \vdots \\ \mu(a_k) + \mu^0(a_k) \end{pmatrix}, \begin{pmatrix} \sigma(a_1, a_1) + \eta^2(a_1) & \ldots & \sigma(a_1, a_k) \\ \vdots & \ddots & \vdots \\ \sigma(a_1, a_k) & \ldots & \sigma(a_k, a_k) + \eta^2(a_k) \end{pmatrix} \right).$$

Following lemma 2.1, $\tau_j(\hat{a}; \mathbf{a})$ is now the $(1, j)^{th}$ entry of matrix

$$\begin{pmatrix} \sigma(a_1, \hat{a}) & \ldots & \sigma(a_k, \hat{a}) \end{pmatrix} \begin{pmatrix} \sigma(a_1, a_1) + \eta^2(a_1) & \sigma(a_1, a_2) & \ldots & \sigma(a_1, a_k) \\ \sigma(a_1, a_2) & \sigma(a_2, a_2) + \eta^2(a_2) & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(a_1, a_k) & \sigma(a_2, a_k) & \ldots & \sigma(a_k, a_k) + \eta^2(a_k) \end{pmatrix}^{-1}.$$

The posterior variance is as in equation (8), where $\tau_j(\mathbf{a})$ is derived from $\tau_j(\hat{a}; \mathbf{a})$ above. As in theorem 3.1(iii), μ^0 enters neither posterior variance nor the single-player sample. □

Example E.1 (Noisier observations, more uncertain attributes)**.** *Let $k = 1$ and consider the Brownian covariance σ_{br} from section F.1. Let $\omega(a) = 1$ for all $a \in [0, 1]$. Suppose observations are of the form $y(a) = f(a) + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \eta^2)$. As expected, for any sample $a \in [0, 1]$ $\psi^2(a)$ is decreasing in noise η^2. The optimal sample $a^*(\eta)$ is pinned down by*

$$a^*(\eta)\,(3a^*(\eta) - 2) - 4\,(1 - a^*(\eta))\,\eta^2 = 0.$$

From section F.1, $a^*(0) = 2/3$. So by implicit differentiation with respect to η:

$$\frac{\partial a^*(\eta)}{\partial \eta} = \frac{4\eta(1 - a^*(\eta))}{3a^*(\eta) + 2\eta^2 - 1} > 0 \quad \text{for } a^* \in (1/3, 1), \eta > 0.$$

Therefore, the higher η^2 is, the further away the single-player sample is from the benchmark attribute $\bar{a} = 0$. In the presence of greater observational noise, the player seeks attributes that are ex ante more uncertain.

E.4 A violation of assumption 2

Consider linear covariance $\sigma_{lin}(a, a') = 1 + aa'$, $\omega(a) = 1$ for any $a \in [0, 1]$, and $k = 1$. For any $\mathbf{a} = (a)$,

$$\tau(\mathbf{a}) = \int_0^1 \tau(\hat{a}; \mathbf{a})\, \mathrm{d}a = \int_0^1 \frac{1 + a\hat{a}}{1 + a^2}\, \mathrm{d}\hat{a} = \frac{2 + a}{2(1 + a^2)}.$$

It is immediate then that the posterior variance is $\psi^2(\mathbf{a}) = (2 + a)^2/4(1 + a^2)$. This posterior variance is maximized at $a^* = 1/2$, with corresponding $\psi^2(1/2) = 5/4$. But note that $k = 1$ is sufficient to learn the true value of the project because its ex-ante uncertainty is exactly

$$\mathrm{Var}(v) = \int_0^1 \int_0^1 \sigma_{lin}(a, a')\, \mathrm{d}a\, \mathrm{d}a' = 5/4.$$

Note that there is non-zero residual uncertainty about individual attributes after sampling a^*: the ex-post variance of $f(\hat{a})$ is $(\hat{a} - a^*)^2/(1 + a^{*2})$.

F Other examples of single-player sampling

F.1 Brownian covariance

This example considers a covariance function that naturally encodes the presence of prior knowledge about some attributes. A *benchmark attribute* is an attribute $\bar{a} \in \mathcal{A} = [0, 1]$ such that (i) $f(\bar{a})$ is ex ante known, i.e. $\sigma(\bar{a}, \bar{a}) = 0$, (ii) $f(\bar{a})$ determines the player's expectation for all other attribute realizations, and (iii) the variance of each attribute realization $f(a)$ is proportional to a's distance from $\bar{a}$. That is, the covariance is given by

$$\sigma_{\bar{a}}(a, a') = \begin{cases} \min(a - \bar{a}, a' - \bar{a}) & \text{if } a, a' \geq \bar{a} \\ \min(\bar{a} - a, \bar{a} - a') & \text{if } a, a' \leq \bar{a} \\ 0 & \text{if } a < \bar{a} < a' \text{ or } a' < \bar{a} < a. \end{cases} \tag{1}$$

This covariance function corresponds to f being a Brownian path.[1] Figure 2a illustrates

[1] To be precise, attribute mappings that result from $\sigma_{\bar{a}}$ follow two independent Brownian motions $X_1(a - \bar{a})$

that if the attribute mapping is a Brownian path extrapolation from a sample is linear and traverses the benchmark realization $f(\bar{a})$.

Given sample $\mathbf{a}$, let $\bar{\mathbf{a}} = (a_1, \ldots, a_{k+1})$ denote the *expanded sample* that consists of $\mathbf{a}$ and $\bar{a}$, ordered and reindexed so that $a_1 < \ldots < a_{k+1}$. Effectively, the presence of a benchmark attribute gives an attribute for free to the player.

Lemma F.1. *Fix $\bar{\mathbf{a}} = (a_1, \ldots, a_{k+1})$ so that $0 \leq a_1 < \ldots < a_{k+1} \leq 1$. Sample realization $f(a_j)$ is weighted by*

$$\tau_j(\bar{\mathbf{a}}) = \begin{cases} (a_1 + a_2)/2 & \text{if } j = 1 \\ (a_{j+1} - a_{j-1})/(2) & \text{if } j \in \{2, \ldots, k\} \\ (a_k + a_{k+1})/2 & \text{if } j = k+1. \end{cases}$$

Proof for lemma F.1. Without loss, we let $\bar{a} = 0$ and $\mu(a) = 0$ for all $a \in [0, 1]$. Hence, $f(\bar{a}) = 0$ as well. We first characterize the precision matrix for any sample $\mathbf{a}$.

Claim 1. *Fix $\mathbf{a} = (a_1, \ldots, a_k)$ such that $0 < a_1 < \ldots < a_k \leq 1$. The $(i, j)^{th}$ entry of the sample precision matrix $P(\mathbf{a}) := [\Sigma(\mathbf{a})]^{-1}$ is given by*

$$p_{ij} = \begin{cases} \dfrac{1}{a_i - a_{i-1}} + \dfrac{1}{a_{i+1} - a_i} & \text{for } j = i \\ -\dfrac{1}{|a_j - a_i|} & \text{for } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

where $a_0 = 0$ and $a_{k+1} = +\infty$.

Proof. Fix $\mathbf{a} = (a_1, \ldots, a_k)$. With some abuse of notation, let $f(\mathbf{a} \setminus B)$ denote the vector of realizations for all sample attributes other than those in B. We use two facts: (i) p_{ii} is the the reciprocal of the conditional variance of $f(a_i)$ given $f(\mathbf{a} \setminus a_i)$, and (ii) p_{ij} is the conditional covariance of $f(a_i)$ and $f(a_j)$ given $f(\mathbf{a} \setminus \{a_i, a_j\})$ (Theorem 2.3 in Rue and Held (2005)). So

$$\begin{aligned} \text{Var}\left(f(a_i) \mid f(\mathbf{a} \setminus a_i)\right) &= \frac{(a_i - a_{i-1})(a_{i+1} - a_i)}{a_{i+1} - a_{i-1}} \\ \Rightarrow \qquad p_{ii} &= \frac{1}{\text{Var}\left(f(a_i) \mid f(\mathbf{a} \setminus a_i)\right)} = \frac{1}{a_i - a_{i-1}} + \frac{1}{a_{i+1} - a_i}. \end{aligned}$$

where $a_0 = 0$ and $a_{k+1} = +\infty$. This gives us the diagonal entries of $\Sigma^{-1}(\mathbf{a})$. For off-diagonal

and $X_2(\bar{a} - a)$ defined on domains $[\bar{a}, 1]$ and $[0, \bar{a}]$ respectively. As in Callander (2011), attribute realizations between any two known sample realizations follow a Brownian bridge.

terms, for any $a_j, a_i \in \mathbf{a}$

$$\mathrm{Cov}\left(f(a_i), f(a_j) \mid f(\mathbf{a} \setminus \{a_i, a_j\})\right) = -\frac{p_{ij}}{p_{ii}p_{jj} - p_{ij}^2}.$$

By the Markov property of the Brownian motion, $f(a_{i+1})$ and $f(a_{i-1})$ are independent conditional on $f(a_i)$ for any i. Hence, $p_{ij} = 0$ for any i, j such that $|i - j| > 1$. Consider first a_{k-1} and a_k:

$$\mathrm{Cov}\left(f(a_{k-1}), f(a_k) \mid f(\mathbf{a} \setminus \{a_{k-1}, a_k\})\right) = a_{k-1} - a_{k-2} \quad \Rightarrow \quad p_{k-1,k} = -\frac{1}{a_k - a_{k-1}}.$$

There are two solutions for $p_{k-1,k}$, one of which is the one above and the other is $p_{k-1,k} = -p_{k-1,k-1}$. In order for $\Sigma^{-1}(\mathbf{a})$ to be non-singular, and hence invertible, we need that $p_{ii}p_{jj} - p_{ij}^2 \geq 0$. The latter solution is therefore inadmissible. For any $i < k$,

$$\mathrm{Cov}\left(f(a_{i-1}), f(a_i) \mid f(\mathbf{a} \setminus \{a_{i-1}, a_i\})\right) = \frac{(a_{i-1} - a_{i-2})(a_{i+1} - a_i)}{a_{i+1} - a_{i-2}}$$

applying the formula for the conditional covariance of the multivariate normal distribution of the sample. Therefore, the two solutions for $p_{i-1,i}$ are:

$$p_{i-1,i}^* = -\frac{1}{a_i - a_{i-1}}, \quad p_{i-1,i}^{**} = -\frac{(a_i - a_{i-2})(a_{i+1} - a_{i-1})}{(a_{i+1} - a_i)(a_i - a_{i-1})(a_{i-1} - a_{i-2})}.$$

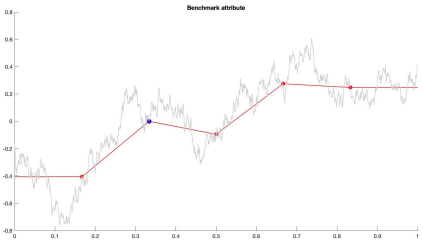
Of the two solutions, again, only $p_{i-1,i}^*$ is admissible because it is the only one to guarantee that $\Sigma^{-1}(\mathbf{a})$ is non-singular. □

Take $a \in (a_i, a_{i+1})$ for $i = 0, \ldots, k-1$. The weight given to attribute realization $f(a_i)$ is

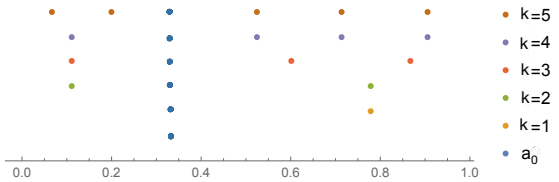
$$\begin{aligned}
\tau_i(a; \mathbf{a}) &= \sigma(a_i, a)p_{ii} + \sum_{j \neq i} p_{ij}\sigma(a_j, a) \\
&= \sigma(a_i, a)p_{ii} + \sigma(a, a_{i-1})p_{i-1,i} + \sigma(a_{i+1}, a)p_{i,i+1} \\
&= a_i\left(\frac{1}{a_i - a_{i-1}} + \frac{1}{a_{i+1} - a_i}\right) - a_{i-1}\frac{1}{a_i - a_{i-1}} - a\frac{1}{a_{i+1} - a_i} \\
&= \frac{a_{i+1} - a}{a_{i+1} - a_i}.
\end{aligned}$$

From a similar calculation, $\tau_{i+1}(a; \mathbf{a}) = (a - a_i)/(a_{i+1} - a_i)$ and $\tau_j(a; \mathbf{a}) = 0$ for any $j \neq i, i+1$. For $a \in (a_k, 1)$, $\tau_k(a; \mathbf{a}) = 1$ and $\tau_j(a; \mathbf{a}) = 0$ for all $j \neq k$. Integrating as in equation (6) in lemma 2.2, we obtain $\tau_j(\mathbf{a})$ for all $j = 1, \ldots, k$. □

Lemma F.1 makes precise the local nature of extrapolation that characterizes the Brownian covariance. An interior attribute $a_j \in \bar{\mathbf{a}}$ is useful only insofar as it is informative about attributes to its immediate left and right in $[a_{j-1}, a_j]$ and $[a_j, a_{j+1}]$. Therefore, when all attributes are equally important to the player, realization $f(a_j)$ is weighted proportionally to the mass of attributes it is informative about. Note, also, that $f(a_1)$ and $f(a_{k+1})$ are the only sources of information for peripheral attributes in $[0, a_1]$ and $[a_{k+1}, 1]$ respectively.



(a) The true mapping is in grey and the extrapolated mapping in red. Sample attributes: $\mathbf{a} = (1/6, 3/6, 4/6, 5/6)$.



(b) Single-player sample for $k \in \{1, \ldots, 5\}$

Figure 2: Benchmark attribute $\bar{a} = 1/3, f(\bar{a}) = 0$, $\sigma_{1/3}$ *and* $\mu(a) = 0$ *for all* $a \in [0, 1]$.

Proposition F.2 characterizes the optimal sample, illustrated in Figure 2b for $\bar{a} = 1/3$. The single-player sample is not equally spaced in $[0, \bar{a}]$ and $[\bar{a}, 1]$: attributes are systematically closer to peripheral attributes $a = 0$ and $a = 1$ to account for the fact that $f(\bar{a})$ is known. Moreover, as capacity increases from k to $k' > k$, the single-player sample for k' has weakly more attributes on each side of $\bar{a}$ than that for k.[2]

Proposition F.2 (Quasi-representative sampling)**.** *The single-player sample* $\mathbf{a}^* = (a_1^*, \ldots, a_k^*)$, *where* $a_1^* < \ldots < a_m^* < \bar{a} < a_{m+1}^* < \ldots < a_k^*$, *consists of*

$$a_j^* = \begin{cases} \bar{a} - \dfrac{2(m-j+1)}{2m+1}\bar{a} & \text{if } \; j \leq m \\ \bar{a} + \dfrac{2(j-m)}{2(k-m)+1}(1-\bar{a}) & \text{if } \; j \geq m+1 \end{cases}$$

and $m \in \mathbb{N}$ *solves*

$$\max_{0 \leq m \leq k} \quad \frac{4}{3}\left(\bar{a}^3 \frac{m(m+1)}{(2m+1)^2} + (1-\bar{a})^3 \frac{(k-m)(k-m+1)}{(2(k-m)+1)^2}\right).$$

Proof for proposition F.2. Consider a subsample $\mathbf{a}_n$ of size n in $(\bar{a}, 1]$. Let $\epsilon_j := a_j - a_{j-1}$

[2]Moreover, the single-player sample satisfies a *zig-zag property*: e.g., as the number of attributes in $[0, \bar{a}]$ goes from m to $m+1$ and the single-player sample from $\mathbf{a}$ to $\mathbf{a}'$, $a_j' < a_j$ for $j = 1, \ldots, m$ and $a_{m+1}' \in (a_m, \bar{a})$.

for $j = 1, \ldots, n$. Using lemma F.1, the posterior variance of this subsample is

$$\psi^2(\mathbf{a}_n) = \sum_{j=1}^{n} \epsilon_j \left(1 - \bar{a} - \sum_{h=1}^{j-1} \epsilon_h - \epsilon_j/2\right)^2 =: \sum_{j=1}^{n} \epsilon_j \left(T_j - \epsilon_j/2\right)^2,$$

where T_j denotes the remaining distance from a_j to 1. Taking the first-order condition with respect to ϵ_j for $j = 1, \ldots, n$ and solving backwards from ϵ_n^*, we obtain

$$\epsilon_{n-j}^* = \frac{2}{2j+3} T_{n-j}.$$

Hence $\epsilon_1^* = 2/(2(n-1)+3)(1-\bar{a})$ and all distances are equal, i.e. $\epsilon_j^* = 2/(2n+1)(1-\bar{a})$. Therefore, the constrained-optimal sample $\mathbf{a}_n^* = (a_1^*, \ldots, a_n^*)$ is given by

$$a_j^* = \bar{a} + \frac{2j}{2n+1}(1-\bar{a}).$$

Therefore, fixing $n \leq k$ the maximal posterior variance on $[\bar{a}, 1]$ is

$$\psi^2(\mathbf{a}_n^*) = \frac{4}{3}(1-\bar{a})^3 \frac{n(n+1)}{(2n+1)^2}.$$

Due to the fact that the Brownian motions on $[0, \bar{a}]$ and $[\bar{a}, 1]$ are independent, the optimal sample $\mathbf{a}^*$ on $[0, 1]$ solves

$$\max_{0 \leq m \leq k, m \in \mathbb{N}} \quad \frac{4}{3}\left(\bar{a}^3 \frac{m(m+1)}{(2m+1)^2} + (1-\bar{a})^3 \frac{(k-m)(k-m+1)}{(2(k-m)+1)^2}\right).$$

□

This characterization implies that small changes in the benchmark attribute $\bar{a}$ might result in drastically different sampling patterns. Take, for instance, two benchmark attributes $\bar{a} = 1/2 - \epsilon$ and $\bar{a}' = 1/2 + \epsilon$ for $\epsilon > 0$ arbitrarily small and $k = 1$. As ϵ shrinks to zero $a^*(\bar{a}) \to 5/6$ but $a^*(\bar{a}') \to 1/6$. That is, two ex-ante comparable projects are evaluated based on very different sets of attributes.

Lastly, it is worth emphasizing that this exercise could easily incorporate prior uncertainty about the benchmark, e.g. $f(\bar{a}) \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$ for $\bar{\sigma} > 0$. The variance of any realization $f(a)$ now becomes $\bar{\sigma}^2 + |a - \bar{a}|$. The uncertainty about each attribute is still proportional to that attribute's distance from $\bar{a}$. Therefore single-player sampling is qualitatively the same as in proposition F.2.

F.2 Polynomial covariance

This example considers a simple non-Markovian covariance for which extrapolation is non-local and the project can be fully learned with sufficiently large capacity. Let

$$\sigma_{qd}(a, a') = (1 + aa')^2 \text{ for } a, a' \in [0, 1]. \tag{2}$$

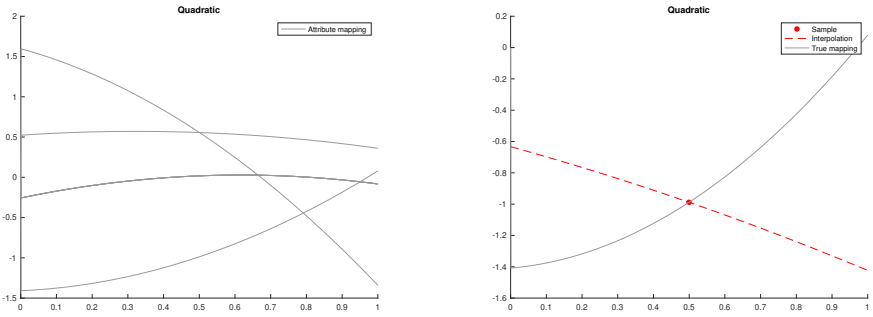
Figure 3a illustrates the shape of attribute mappings that result from this covariance. The total ex-ante uncertainty about the value of a project described by σ_{qd} is:

$$\text{Var}(v) = \int_0^1 \int_0^1 \sigma_{qd}(a, a') \, \mathrm{d}a \, \mathrm{d}a' = \frac{29}{18}.$$

Fix $k = 1$ first. Given a singleton sample $\mathbf{a} = (a)$, $\tau_1(\hat{a}; \mathbf{a}) = (1 + a\hat{a})^2/(1 + a^2)^2$ for any $\hat{a} \in [0, 1]$. The extrapolated mapping is strictly decreasing if and only if $f(a) < 0$ (Figure 3b). The weight given to the sample attribute and the posterior variance are respectively

$$\tau_1(\mathbf{a}) = \frac{3 + a(3 + a)}{3(1 + a^2)^2} > 0, \quad \psi^2(\mathbf{a}) = \frac{\left((1 + a)^3 - 1\right)^2}{9a^2(1 + a^2)^2}.$$

It follows that the optimal sample consists of $a^* = \left(\sqrt{13} - 2\right)/3 \approx 0.54$, attaining $\psi^2(a^*) \approx 1.607$. Strikingly, the player's ability to learn a single attribute is very valuable, as its realization removes about 99.7% of the ex-ante uncertainty about the project.



(a) Possible attribute mappings *(b) Singleton sample* $\mathbf{a} = (1/2)$

Figure 3: Quadratic covariance $\sigma_{qd}(a, a') = (1 + aa')^2$ *and* $\mu(a) = 0$ *for all* $a, a' \in [0, 1]$

A similar calculation for $k = 2$ – collected in lemma F.3 – reveals that the optimal two-attribute sample consists of $a_1^* = 5/32$ and $a_2^* = 49/66$. The posterior variance that this sample attains is $29/18$, which is exactly equal to the ex-ante uncertainty about the project $\text{Var}(v)$. Therefore, the area under the parabolic attribute mapping resulting from σ_{qd} can be fully learned from the realizations of two optimally selected attributes. The sample $\mathbf{a}^* = (5/32, 49/66)$ is optimal for any $k > 2$ as well.

Lemma F.3. *Fix polynomial covariance $\sigma_{qd}(a, a') = (1 + aa')^2$ for all $a, a' \in [0, 1]$ and sample $\mathbf{a} = (a_1, a_2)$ s.t. $a_1 < a_2$. Then, $f(a_1)$ and $f(a_2)$ are weighted respectively by*

$$\tau_1(\mathbf{a}) = \frac{3 + a_1 - 5a_2 - a_1 a_2^2 - 3(1 + a_1)a_2^3}{3(a_1 - a_2)(2 + 2a_1 a_2 + a_2^2 + a_1^2(1 + 2a_2^2))},$$
$$\tau_2(\mathbf{a}) = -\frac{3 + a_2 - 5a_1 - a_2 a_1^2 - 3(1 + a_2)a_1^3}{3(a_1 - a_2)(2 + 2a_1 a_2 + a_2^2 + a_1^2(1 + 2a_2^2))}.$$

The posterior variance of $\mathbf{a}$ is

$$\psi^2(\mathbf{a}) = \frac{27 + 2a_2(3 + 5a_2) + 2a_1(3 + a_2(4 + 9a_2)) + a_1^2(10 + a_2(18 + 11a_2))}{9(2 + 2a_1 a_2 + a_2^2 + a_1^2(1 + 2a_2^2))}$$

and it is maximized at $\mathbf{a}^* = (5/32, 49/66)$, with corresponding $\psi^2(\mathbf{a}^*) = 29/18$.

Proof. From a direct application of lemma 2.1, we obtain

$$\tau_1(\hat{a}; \mathbf{a}) = \frac{a_2 - \hat{a}}{a_2 - a_1} \frac{2 + a_2(a_2 + \hat{a}) + a_1(a_2 + \hat{a} + 2a_2^2 \hat{a})}{2 + 2a_1 a_2 + a_2^2 + a_1^2(1 + 2a_2^2)},$$

$$\tau_2(\hat{a}; \mathbf{a}) = \frac{\hat{a} - a_1}{a_2 - a_1} \frac{2 + a_1(a_1 + a_2) + \hat{a}(a_1 + a_2 + 2a_1^2 a_2)}{2 + 2a_1 a_2 + a_2^2 + a_1^2(1 + 2a_2^2)}.$$

By integrating with respect to $\hat{a}$, we obtain $\tau_1(\mathbf{a})$ and $\tau_2(\mathbf{a})$. Posterior variance $\psi^2(\mathbf{a})$ follows from a direct application of (8). □

G Supplementary material for section 5 (Site selection)

G.1 Formal results and proofs on strategic site selection

To reiterate, the evaluator's prior value is the expected realization at his site $\nu_0^P = \mu(a_P) := \mu_P$. The researcher's prior value is the average expected outcome

$$\nu_0^A = \int_0^1 \mu(a)\, \mathrm{d}a =: \bar{\mu}.$$

Because any singleton site correlates the players' posterior values perfectly, the sufficient statistics pair corresponding to $a \in [0, 1]$ simplifies to $(\alpha_1(a), \alpha_2(a)) = (\psi_P(a), \psi_A(a))$. Each site is described by the pair of posterior variances it induces. Figure 4 plots the single-peaked sufficient statistics and the corresponding single-player optima.

Optimal selection of a compromise site. We refer to the interval $[1/2, a_P]$ as *compromise sites*: slightly shifting the pilot site within this interval strictly increases informativeness for one player and decreases it for the other. As Figure 4 depicts, ψ_A is strictly decreasing and

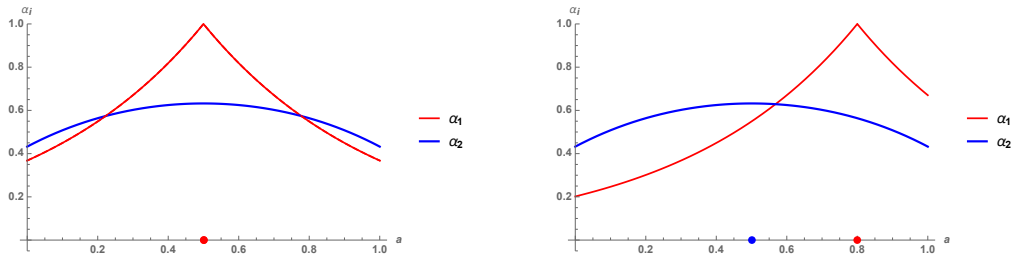


Figure 4: The plots show $(\alpha_1(\cdot), \alpha_2(\cdot))$ for $a_P = 1/2$ (left) and $a_P = 4/5$ (right) when $\ell = 1/2$. The blue and red dots show single-agent site and single-principal site respectively.

ψ_P is strictly increasing over the compromise region, so there is a sharp tradeoff between the two sufficient statistics. If $a_P = 1/2$, the compromise interval collapses to the median site.

Let $a^*(\mu_P, \bar{\mu})$ denote the optimal site given players' prior values. Corollary G.1, which follows from proposition 4.5, shows that prior disagreement always leads to the optimal selection of a compromise site. That is because – all else fixed – under prior disagreement the researcher's payoff is increasing in ψ_P: higher variance in the evaluator's decision increases the likelihood that it coincides with the decision suggested by $\bar{\mu}$. At any site that is not a compromise, ψ_A and ψ_P are either both strictly increasing or both strictly decreasing. The exact tradeoff between ψ_A and ψ_P within the compromise region – and hence $a^*(\mu_P, \bar{\mu})$ – is generically determined by the relative magnitudes of μ_P and $\bar{\mu}$. In the special case of $a_P = 1/2$, the optimal site is exactly the median site.

Corollary G.1 (Compromise site selection)**.**

(i) For $\mu_P \lesseqgtr 0$ and $\bar{\mu} \gtreqless 0$, the optimal site is a compromise: $a^*(\mu_P, \bar{\mu}) \in [1/2, a_P]$.

(ii) For any $\bar{\mu} \in \mathbb{R}$, $a^*(0, \bar{\mu}) = 1/2$.

(iii) For $\bar{\mu} = 0$, the optimal site $a^*(\mu_P, 0)$ is increasing in $|\mu_P|$ and $a^*(\mu_P, 0) = a^*(-\mu_P, 0)$ for any μ_P. For sufficiently large $|\mu_P|$, $a^*(\mu_P, 0) = a_P$.

Proof of corollary G.1. We first show that posterior variances are single-peaked in a. For any $a \in \mathcal{A}$, $\psi_A(a) = \ell\left(2 - e^{-(1-a)/\ell} - e^{-a/\ell}\right)$ and $\psi_P(a) = e^{-|a-a_P|/\ell}$. It is immediate that ψ_A is strictly increasing for $a < 1/2$ and strictly decreasing for $a > 1/2$. On the other hand, ψ_P is strictly increasing for $a < a_P$ and strictly decreasing for $a > a_P$. The rest of the argument follows from proposition 4.5 and lemma C.1. □

Optimal selection of a peripheral site. The possibility of an optimal site that is not a compromise arises if players are in ex ante agreement. Figure 6b plots the unique optimal site as a function of μ_P. For $|\mu_P|$ sufficiently close to zero, the optimal site is to the left of the median site and hence further away from a_P than the median site. The researcher undershoots his preferred site to influence a skeptical evaluator. As $|\mu_P|$ increases, the optimum returns

into the compromise region. For sufficiently high $|\mu_P|$ the optimal site is a_P. Corollary G.2 establishes that this pattern of site selection holds generally. In fact, it holds even when the median site is the most informative site for both players.

Corollary G.2 (Peripheral site selection)**.** *Suppose that players are in ex ante agreement.*

(i) For $a_P > 1/2$ and μ_P sufficiently close to zero, $a^*(\mu_P, \bar{\mu}) < 1/2$.

(ii) For $a_P = 1/2$ and μ_P sufficiently close to zero, there are exactly two optimal sites and they are equidistant from the median site.

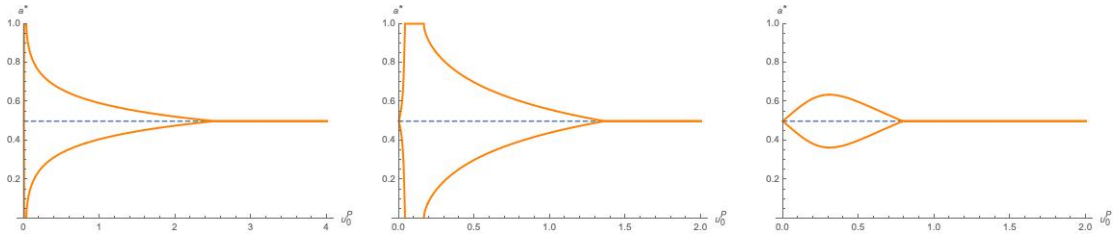
Proof. (i) Without loss, let $\bar{\mu} > 0$. The researcher's payoff is strictly decreasing at $a = 1/2$ iff

$$\mu_P \left(-e^{-1/\ell} \bar{\mu} + 2e^{a_P/\ell} \left(e^{1/(2\ell)} - 1 \right) \ell \mu_P \right) < 0.$$

Because $\bar{\mu} > 0$, this inequality holds for $\mu_P > 0$ close to zero. But from proposition G.1, $a^*(0, \bar{\mu}) = 1/2$ for any $\bar{\mu}$. Hence, by the continuity of the optimal site in $(\mu_P, \bar{\mu})$, $a^*(\mu_P, \bar{\mu}) < 1/2$ for μ_P sufficiently close to zero. The optimal site solves

$$e^{(1+2a)/\ell} - e^{4a/\ell} = \frac{e^{(1+2a+a_P)/\ell}}{\ell} \bar{\mu} \mu_P + e^{2a_P} \ell \left(e^{1/\ell} + e^{2a/\ell} - 2e^{(1+a)/\ell} \right) {\mu_P}^2. \tag{3}$$

(ii) Let $a_P = 1/2$. It is straightforward to verify that if equation (3) holds for $a = 1/2 + \epsilon$ then it also holds for $a' = 1/2 - \epsilon$, where $\epsilon \in (0, 1/2)$. Moreover, the researcher's payoff has a strict local minimum at the median site if and only if $\mu(1/2) \left(2\ell \mu(1/2) \left(e^{1/(2\ell)} - 1 \right) - e^{1/(2\ell)} \bar{\mu} \right) < 0$ which holds if (i) $\mu_P = \mu(1/2)$ and $\bar{\mu}$ have the same sign, and (ii) $|\mu(1/2)|$ is sufficiently small. □



(a) $\ell = 1/10$ *(b) $\ell = 1/5$* *(c) $\ell = 1/2$*

Figure 5: Optimal site for $a_P = 1/2$, $\bar{\mu} = 1/2$, $r_A = r_P = 0$, and $\ell \in \{1/10, 1/5, 1/2\}$.

Distortions are largest for moderate correlation. Figure 5 illustrates optimal site selection for $a_P = 1/2$: due to the symmetry of the Ornstein-Uhlenbeck process, there are at most two optimal sites symmetric around the median site. As correlation across sites becomes either arbitrarily strong ($\ell \to +\infty$) or arbitrarily weak ($\ell \to 0$) distortions vanish. For any $\mu_P \in \mathbb{R}$, the optimal site converges to the median site. Site selection is no longer an

effective instrument for influencing the evaluator's behavior. On the other hand, Figure 5b illustrates an instance in which moderate correlation leads to selection of the most peripheral sites $\{0, 1\}$. These are the *least informative sites* for both the researcher and the evaluator, yet the researcher prefers these sites relative to forgoing pilot evaluations altogether.

G.2 Comparison to random site selection

A natural alternative to purposive site selection is random selection of pilot sites. Yet Olsen et al. (2013) have observed that the latter is rare in social experiments. Purposive sampling – purported to generate a more diverse sample than uniform random sampling – is the preferred approach in practice. They show that of the 273 impact evaluations included in the *Digest of Social Experiments* (as compiled in Greenberg and Shroder (2004)), all but 7 featured purposive site selection. Motivated by this fact, we compare purposive site selection to random selection by a researcher who weighs all sites equally.

Two empirically common random formats for sampling sites are considered – uniform sampling and stratified uniform sampling – both of which do not make use of covariance σ_{ou} and are suboptimal by theorem 3.1.[3] The former selects k sites randomly from $[0, 1]$, whereas the latter divides $[0, 1]$ into k equal-size bins before selecting a site randomly from each bin. Lemma G.3 derives the expected sample for each format. For instance, for $k = 2$ the expected uniform sample is $(1/3, 2/3)$, whereas the expected stratified one is $(1/4, 3/4)$.

Lemma G.3 (Random site selection)**.**

(i) (Uniform sample) Suppose $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ are drawn independently from $U(0, 1)$, indexed so that $\bar{a}_1 < \bar{a}_2 < \ldots < \bar{a}_k$. Then $\mathbb{E}[\bar{a}_j] = j/(k+1)$ for any $j = 1, \ldots, k$.

(ii) (Stratified uniform sample) Suppose $\underline{a}_j$ is drawn from $U[(j-1)/k, j/k]$ for $j = 1, \ldots, k$. Then, $\mathbb{E}[\underline{a}_j] = (2j-1)/(2k)$.

Proof. (i) The density of the j^{th}-order statistic of a sample of k iid observations from a continuous cdf F (with density f) is:

$$f_j(a) = \frac{k!}{(k-j)!(j-1)!} F^{j-1}(a)(1 - F(a))^{k-j} f(a).$$

For the uniform distribution with $F(a) = a$ and $f(a) = 1$, this simplifies to

$$f_j(a) = \frac{k!}{(k-j)!(j-1)!} a^{j-1}(1-a)^{k-j} = \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(k+1-j)} a^{j-1}(1-a)^{(k+1-j)-1}$$

[3] E.g., the evaluation process of the Job Training Partnership Act of 1982 initially aimed to select sites in which to conduct the study through either random or stratified random selection "in order to obtain nationally representative results." See Hotz (1992).

which is the pdf of a Beta distribution with parameters $(j, k+1-j)$. Its mean is therefore $j/(k+1-j+j) = j/(k+1)$.
(ii) The mean of a draw from $U[(j-1)/k, j/k]$ is $(j-1/2)/k = (2j-1)/(2k)$. □

Proposition 3.4 in our main analysis already established that the optimal sample is symmetric around the median site and any two adjacent sites are equidistant. Proposition G.4 sharpens this characterization by establishing monotonicity and convergence of the optimal sites. First, the proposition shows that as ℓ increases – i.e. as sites become more similar in terms of program outcomes and the planner can extrapolate over a longer distance – optimal sites are farther away from the median site and from each other. The set of pilot sites becomes more diverse. The second part of the proposition asserts that the optimal sample of sites converges to the expected random samples described in lemma G.3 as sites become arbitrarily correlated or independent. Effectively, the expected random samples serve as upper and lower bounds for the optimal sample.

Proposition G.4 (Monotonicity and convergence to random site selection)**.**

(i) For any $j < (k+1)/2$ (resp., $j > (k+1)/2$), $a_j^*(\ell)$ is decreasing (increasing) in ℓ.

(ii) As site outcomes approach independence, the single-player sample converges to the expected uniform sample, i.e. as $\ell \to 0$, $a_j^*(\ell) \to \bar{a}_j$ for any $j = 1, \ldots, k$.

(iii) As site outcomes approach perfect correlation, the single-player sample converges to the expected stratified sample, i.e. as $\ell \to +\infty$, $a_j^*(\ell) \to \underline{a}_j$ for any $j = 1, \ldots, k$.

Proof. (i) By proposition 3.4, it is sufficient to establish the monotonicity of a_1^* with respect to ℓ. By implicit differentiation of the equation

$$1 - e^{-a_1^*(\ell)/\ell} = \tanh\left(\frac{1 - 2a_1^*(\ell)}{2\ell(k-1)}\right)$$

with respect to ℓ and straightforward algebraic simplifications, we obtain

$$\frac{\partial a_1^*(\ell)}{\partial \ell} = \frac{1}{2\ell}\left(2a_1^*(\ell) - 1 + \frac{k-1}{k + \tanh\left(\frac{1-2a_1^*(\ell)}{2\ell(k-1)}\right)}\right) < 0.$$

For any $k \geq 2$, $a_1^*(\ell) < 1/2$ satisfies the first-order condition and therefore $\partial a_1^*(\ell)/\partial \ell < 0$. Hence a_1^* is strictly increasing in ℓ for any $k \geq 2$.

(ii) We want to show that as $\ell \to 0$, $a_1^* \to 1/(k+1)$. Substituting the standard trigonometric identity $\frac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}$ for $x \in \mathbb{R}$ into the first-order condition with respect to a_1^*, we obtain $2 - e^{-a_1^*/\ell} - e^{\frac{1-a_1^*(k+1)}{\ell(k-1)}} = 0$. Note that $a_1^* < 1/(k+1)$ because $\left(1 - e^{-a_1/\ell} - \tanh\left(\frac{1-2a_1}{2\ell(k-1)}\right)\right)$ is strictly increasing in a_1 and

$$\left(1 - e^{-a_1/\ell} - \tanh\left(\frac{1-2a_1}{2\ell(k-1)}\right)\right)\Bigg|_{a_1=1/(k+1)} > 0.$$

Because $e^{-a_1^*/\ell} \to 0$, for the FOC to hold even as $\ell \to 0$ we need $e^{\frac{1-a_1^*(k+1)}{\ell(k-1)}} \to 2$. Because $1 - a_1^*(k+1) > 0$, this implies that $1 - a_1^*(k+1) \to 0$ as $\ell \to 0$. Hence, $a_1^* \to 1/(k+1)$ as $\ell \to 0$. Therefore, $a_j^* \to \bar{a}_j$ for any $j = 1, \ldots, k$ as $\ell \to 0$.

(iii) We want to show that as $\ell \to +\infty$, $a_1^* \to 1/(2k)$. By the FOC with respect to a_1^*, we have

$$\lim_{\ell \to +\infty} \quad \frac{1 - e^{-a_1^*/\ell}}{\tanh\left(\frac{1-2a_1^*}{2\ell(k-1)}\right)} = 1.$$

Because the numerator and the denominator each converge to zero as $\ell \to +\infty$, we apply L'Hôpital's rule:

$$\lim_{\ell \to +\infty} \quad \frac{\frac{-a_1^*}{\ell^2} e^{-a_1^*/\ell}}{\frac{1-2a_1^*}{2\ell^2(k-1)} \operatorname{sech}^2\left(\frac{1-2a_1^*}{2\ell(k-1)}\right)} = \lim_{\ell \to +\infty} \quad \frac{-2a_1^*(k-1)}{1-2a_1^*} \frac{e^{-a_1^*/\ell}}{\operatorname{sech}^2\left(\frac{1-2a_1^*}{2\ell(k-1)}\right)} = 1.$$

As $\ell \to +\infty$, $e^{-a_1^*/\ell} \to 1$ and $\operatorname{sech}^2\left(\frac{1-2a_1^*}{2\ell(k-1)}\right) \to 1$. Hence,

$$\lim_{\ell \to +\infty} \quad \frac{-2a_1^*(k-1)}{1-2a_1^*} = 1.$$

This implies that $a_1^* \to 1/(2k)$ as $\ell \to +\infty$. Hence, $a_j^* \to \underline{a}_j$ for any $j = 1, \ldots, k$. □

This result suggests that random site selection with uniform probability leads on average to a too narrow of a sample when sites are correlated. The procedure does not internalize the benefits of local extrapolation. But if sites were perfectly independent ($\ell = 0$), the planner would be indifferent among all sets of k sites, so uniform sampling would perform just as well as any other sampling procedure. On the other hand, stratified sampling is a coarse way of accounting for local extrapolation by separating sites into bins: hence it leads to a too extreme of a sample compared to purposive site selection.

H Supplementary material for section 6 (Sample centrality)

Section H.1 constructs the graph and defines useful objects in it. Section H.2 introduces sample centrality in walk-summable graphs and illustrates it through a simple finite-attribute example. Section H.3 extends the formulation to non-walk-summable graphs, whereas section H.4 discusses the relation of sample centrality to existing centrality measures. Because of our

discussion in section E.2, we consider $\omega(a) \geq 0$ for all $a \in \mathcal{A}$ and $\omega(a)$ is a density over $\mathcal{A}$:

$$\int_{\mathcal{A}} \omega(a)\, \mathrm{d}a = 1.$$

H.1 Graph construction and basic definitions

Graph. Let $\mathcal{G} = (\mathcal{A}, E)$ be an undirected and infinite graph, where $\mathcal{A} := [0,1]^d, d \geq 1$ is the set of attribute-nodes (hereafter referred to as nodes) and E is the set of weighted edges.[4] The weight of an edge aa' joining nodes $a, a' \in \mathcal{A}$ is given by $e_{aa'} := \sigma(a, a')$. Due to unit variances, an edge weight corresponds to the correlation between the two nodes that it joins.[5]

Walks, paths, cycles. A walk w of length $\ell \geq 0$ is a sequence of nodes $w = (a^1, \ldots, a^\ell, a^{\ell+1})$ such that each edge $a^k a^{k+1} \in E$ for $k \in \{1, \ldots, \ell\}$. A path π of length ℓ is a sequence of distinct nodes $\pi = (a^1, \ldots, a^{\ell+1})$ in $\mathcal{A}$. A cycle γ of length ℓ is a sequence of nodes $\gamma = (a^1, \ldots, a^{\ell+1})$ such that $a^1 = a^{\ell+1}$ and nodes $a^1, a^2 \ldots, a^\ell$ are distinct. Let $\kappa(\alpha)$ denote the weight of a walk/path/cycle α of length $\ell(\alpha)$, defined as the product of the weights of the edges it traverses:[6]

$$\kappa(\alpha) := \prod_{m=1}^{\ell(\alpha)} \sigma(a^m, a^{m+1}), \qquad \alpha \in \{w, \pi, \gamma\}.$$

A zero-length walk/path/cycle of the form $\alpha = (a^1)$ has by definition $\kappa(\alpha) = 1$.

Samples of nodes. For $\mathbf{a} = (a_1, \ldots, a_n)$, let $A(\mathbf{a})$ denote the adjacency matrix for subgraph $\mathcal{G}_{\mathbf{a}}$ consisting of nodes $\mathbf{a}$ and edges joining any two nodes in $\mathbf{a}$. The sample covariance matrix is $\Sigma(\mathbf{a}) = I + A(\mathbf{a})$, where I is the $n \times n$ identity matrix. By the Neumann power series for matrix inversion, the precision matrix for the sample can be written as:

$$\Sigma^{-1}(\mathbf{a}) = (I + A(\mathbf{a}))^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell A^\ell(\mathbf{a}).$$

The power series in the right-hand side converges if and only if the spectral radius of matrix $(-A(\mathbf{a}))$, which we denote by $\rho(-A(\mathbf{a}))$, is strictly less than one.[7]

The $(i,j)^{\text{th}}$ entry of $(-A(\mathbf{a}))^\ell$ corresponds to the sum of weights of all ℓ-length walks from a_i to a_j that go exclusively through sample nodes, i.e.

[4] Covariance $\sigma(a, a') = 0$ implies that no edge joins nodes a and a'. In particular, zero-variance attributes correspond to isolated nodes. For ease of exposition, we disregard the presence of such attributes.

[5] The constructed $\mathcal{G}$ differs from a Gauss-Markov random field in that (i) it has uncountably many nodes and (ii) its construction uses covariance σ rather than precision matrices (Rue and Held (2005)).

[6] Note that we define κ here slightly differently from section 6.

[7] The spectral radius of a matrix is given by the maximum absolute value of its eigenvalues. One can easily identify positive definite matrices for which the spectral radius is greater than one.

$$((-1)^\ell A^\ell(\mathbf{a}))_{ij} = \sum_{w: a_i \overset{\ell,\mathbf{a}}{\to} a_j} (-1)^\ell \kappa(w)$$

where $w : a_i \overset{\ell,\mathbf{a}}{\to} a_j$ denotes all walks of length ℓ from a_i to a_j that only visit nodes in $\mathbf{a}$. We refer to

$$(a_i \overset{\mathbf{a}}{\to} a_j) := \sum_{\ell=0}^{\infty} \sum_{w: a_i \overset{\ell,\mathbf{a}}{\to} a_j} (-1)^\ell \kappa(w)$$

as the alternating walk sum from a_i to a_j through sample $\mathbf{a}$.

The alternating walk sum need not always be well-defined, as the walk sum might not converge to the same value for every possible summation order of the walks. In other words, $\mathcal{G}_\mathbf{a}$ might not be walk-summable. Whenever $\mathcal{G}_\mathbf{a}$ is walk-summable, the $(i,j)^{\text{th}}$ entry in $\Sigma^{-1}(\mathbf{a})$ can be expressed as the sum of walks of any length from $a_i \in \mathbf{a}$ to $a_j \in \mathbf{a}$ restricted to go through sample nodes only.

Random subgraph. The constructed graph $\mathcal{G}$ is based only on covariance σ. In particular, it does not use attribute weights ω. In section H.2, we see that ω governs the realization of a random subgraph $\tilde{\mathcal{G}} = (\mathbf{a} \cup a \cup a', E_{\mathbf{a} \cup a \cup a'})$, where (i) nodes a, a' are drawn according to density $\omega : \mathcal{A} \to [0,1]$, (ii) $\mathbf{a}$ is a fixed sample, and (iii) $E_{\mathbf{a} \cup a \cup a'}$ is the set of edges linking nodes in $\mathbf{a} \cup a \cup a'$.

H.2 Walk-summable graph

We first define the walk-summability of $\mathcal{G}$ in increasing order of generality: with respect to a given sample, with respect to all samples of a fixed size, and with respect to samples of any finite size. Our definition of walk-summability is slightly different from others proposed in the literature – e.g. in Malioutov, Johnson and Willsky (2006) – because we need to consider convergence of walk sums in the subgraph of each feasible sample. Subsequently, lemma H.1 provides a necessary and sufficient condition for the walk-summability of $\mathcal{G}$.

Definition H.1 (Walk-summability of $\mathcal{G}$)**.**

(i) Fix $\mathbf{a}$. Graph $\mathcal{G}$ is $\mathbf{a}$-walk-summable if for any $a_i, a_j \in \mathbf{a}$ the alternating walk sum $(a_i \overset{\mathbf{a}}{\to} a_j)$ converges to the same value for every possible summation order of walks.

(ii) For any $n \in \mathbb{N}$, $\mathcal{G}$ is n-walk-summable iff it is $\mathbf{a}$-walk-summable for any n-sample $\mathbf{a}$.

(iii) $\mathcal{G}$ is walk-summable if it is n-walk-summable for any $n \in \mathbb{N}$.

Lemma H.1. *Graph $\mathcal{G}$ is $\mathbf{a}$-walk-summable if and only if $\rho(\bar{A}(\mathbf{a})) < 1$, where $\bar{A}(\mathbf{a})_{ij} := |A(\mathbf{a})_{ij}|$ for any i, j.*

Proof for lemma H.1. Fix a finite sample $\mathbf{a}$ and $a_i, a_j \in \mathbf{a}$. The unordered sum

$$\sum_{\ell} \sum_{w: a_i \overset{\ell, \mathbf{a}}{\to} a_j} (-1)^\ell \kappa(w)$$

converges to the same value despite the order of summation if and only if it converges absolutely, i.e. the sum of absolute terms

$$\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |(-1)^\ell \kappa(w)| = \sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)|$$

converges. The following lemma shows that absolute convergence is equivalent to convergence of $\sum_\ell \bar{A}^\ell(\mathbf{a})$. In turn, convergence of $\sum_\ell \bar{A}^\ell(\mathbf{a})$ is equivalent to $\rho(\bar{A}(\mathbf{a})) < 1$, which gives the desired result.

Lemma H.2. *$\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)|$ converges if and only if $\sum_{\ell=0}^{\infty} \bar{A}^\ell(\mathbf{a})$ converges.*

Proof. ($\Rightarrow$): Since $\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)|$ converges, it also converges absolutely, so the order of summation does not matter. Therefore, it can be rearranged into:

$$\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)| = \sum_{\ell} \sum_{w: a_i \overset{\mathbf{a}, \ell}{\to} a_j} |\kappa(w)| = \sum_{\ell} \bar{A}^\ell(\mathbf{a}).$$

Therefore the RHS converges.
($\Leftarrow$): It is sufficient to show that $\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)|$ converges for a particular ordering of walks. But $\sum_\ell \bar{A}^\ell(\mathbf{a})$ corresponds to one such ordering and it converges, hence by a similar argument to that above, $\sum_{w: a_i \overset{\mathbf{a}}{\to} a_j} |\kappa(w)|$ converges as well. □

□

Next, we unpack $\tau_j(\mathbf{a})$ in terms of walks in the graph. Remember that $\tau_j(\mathbf{a})$ is the weight given to sample observation $f(a_j)$ for $a_j \in \mathbf{a}$ in the agent's posterior. We start with $\tau_j(a; \mathbf{a})$ for some $a \in \mathcal{A}$: the weight given to $f(a_j)$ in the expected value $\mathbb{E}[f(a) \mid \mathbf{a}, f(\mathbf{a})]$. Letting p_{ij} denote the $(i,j)^{th}$ entry of $\Sigma^{-1}(\mathbf{a})$ and $a \in \mathcal{A}$ such that $a \neq a_j$:

$$
\begin{aligned}
\tau_j(a;\mathbf{a}) &= \sigma(a,a_1)p_{1j} + \ldots + \sigma(a,a_n)p_{nj} \\
&= \sigma(a,a_1)(a_1 \overset{\mathbf{a}}{\to} a_j) + \ldots + \sigma(a,a_n)(a_n \overset{\mathbf{a}}{\to} a_j) \\
&= \sum_{a_i \in \mathbf{a}} \sigma(a,a_i)(a_i \overset{\mathbf{a}}{\to} a_j) \\
&= \sum_{a_i \in \mathbf{a}} \sigma(a,a_i) \left(\sum_{\ell=0}^{\infty} \sum_{w: a_i \overset{\ell,\mathbf{a}}{\to} a_j} (-1)^{\ell} \kappa(w) \right) \\
&= \sum_{\ell=1}^{\infty} \sum_{w: a \overset{\ell,\mathbf{a}}{\to} a_j} (-1)^{\ell-1} \kappa(w)
\end{aligned}
\tag{4}
$$

The last line corresponds to all walks from $a \in \mathcal{A}$ to $a_j \in \mathbf{a}$ such that the walk traverses only nodes in $\mathbf{a}$ from the second node and after. A walk of length ℓ has weight $(-1)^{\ell-1}$ rather than $(-1)^{\ell}$.

Lemma H.3 (Graph representation of $\tau(\mathbf{a})$)**.** *Fix $\mathbf{a} = (a_1, \ldots, a_n)$. The weight $\tau_j(\mathbf{a})$ given to observation $f(a_j)$ for $j = 1, \ldots, n$ corresponds to the expected alternating walk sum from node a drawn randomly according to density $\omega : \mathcal{A} \to [0,1]$ to node $a_j \in \mathbf{a}$ such that in all walks from a to a_j all nodes except possibly the first one are in $\mathbf{a}$.*

Proof. By the definition of $\tau_j(\mathbf{a})$ and the expression for $\tau_j(a;\mathbf{a})$,

$$
\tau_j(\mathbf{a}) = \int_{\mathcal{A}} \tau_j(a;\mathbf{a})\omega(a)\,\mathrm{d}a = \int_{\mathcal{A}} \left(\sum_{\ell=1}^{\infty} \sum_{w: a \overset{\ell,\mathbf{a}}{\to} a_j} (-1)^{\ell-1} \kappa(w) \right) \omega(a)\,\mathrm{d}a := -\int_{\mathcal{A}} (a \overset{\mathbf{a}}{\to} a_j)\omega(a)\,\mathrm{d}a
$$

That is, fixing a sample $\mathbf{a}$: (i) a random subgraph $\mathcal{G}' := \{a\} \cup \mathbf{a}$ is drawn, (ii) we compute the expected sum of walks from a to a_j that only traverse sample nodes and obtain $\tau_j(\mathbf{a})$ as above. □

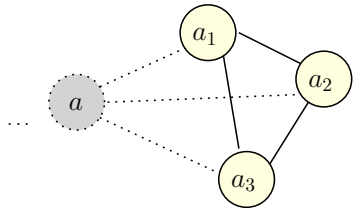


Figure 6: Construction in lemma H.3. Sample nodes $\mathbf{a} = (a_1, a_2, a_3)$ are shown in yellow, and randomly drawn node a in grey.

Because the attribute space is infinite, if ω has an interval support the randomly drawn

starting node a in lemma H.3 is almost surely not in $\mathbf{a}$. Therefore, $\tau_j(\mathbf{a})$ quantifies the distance in the graph from out-of-sample nodes to each node in sample $\mathbf{a}$. Figure 6 depicts the construction in lemma H.3.

Proof for theorem 6.1. (i) The posterior variance $\psi^2(\mathbf{a})$ can be rewritten as

$$\begin{aligned}
\psi^2(\mathbf{a}) &= \sum_{a_i \in \mathbf{a}} \sum_{a_j \in \mathbf{a}} \tau_i(\mathbf{a}) \tau_j(\mathbf{a}) \sigma(a_i, a_j) \\
&= \sum_{a_i \in \mathbf{a}} \sum_{a_j \in \mathbf{a}} \left(- \int_{\mathcal{A}} (a' \overset{\mathbf{a}}{\to} a_i) \omega(a') \, \mathrm{d}a' \right) \left(- \int_{\mathcal{A}} (a'' \overset{\mathbf{a}}{\to} a_j) \omega(a'') \, \mathrm{d}a'' \right) \sigma(a_i, a_j) \\
&= \sum_{a_i \in \mathbf{a}} \sum_{a_j \in \mathbf{a}} \mathbb{E}_{a'} \left[a' \overset{\mathbf{a}}{\to} a_i \right] \sigma(a_i, a_j) \mathbb{E}_{a''} \left[a_j \overset{\mathbf{a}}{\to} a'' \right] \\
&= \int_{\mathcal{A}} \int_{\mathcal{A}} \left(a' \overset{\mathbf{a}}{\to} a'' \right) \omega(a') \omega(a'') \, \mathrm{d}a' \, \mathrm{d}a'' \\
&= \mathbb{E}_{(a,a')} \left[a' \overset{\mathbf{a}}{\to} a'' \right] = \gamma(\mathbf{a}),
\end{aligned}$$

where the second equality follows from lemma H.3, the third uses the notation for expected walk sums, the fourth takes the expectation over the first and last node of each walk in the walk sum outside the summation, and the last line applies Definition 4.

(ii) If $\mathcal{G}$ is k-walk-summable, then it is $\mathbf{a}$-walk-summable for any k-sample $\mathbf{a}$. Hence, by part (i) and theorem 3.1, any single-player sample attains the highest sample centrality. □

Remark H.1 (Finite $\mathcal{A}$)**.** *A natural question is how sample centrality would simplify if the total number of attribute-nodes is finite. If $\mathcal{A}$ is finite, the randomly drawn nodes a, a' in Definition 4 are (generically) sample nodes with positive probability. Our interpretation of $\tau_j(\mathbf{a})$ has to be modified accordingly. Without loss, let $\mathcal{A} := \{a_1, \ldots, a_N\}$ and consider sample $\mathbf{a} := (a_1, a_2)$. By an argument similar to that of lemma H.3,*

$$\begin{aligned}
\tau_1(\mathbf{a}) &= \omega(a_1) + \omega(a_3)(a_3 \overset{\mathbf{a}}{\to} a_1) + \ldots + \omega(a_N)(a_N \overset{\mathbf{a}}{\to} a_1) \\
&= \Pr(a_1)(a_1 \overset{\ell=0}{\to} a_1) + \Pr(a_3)(a_3 \overset{\mathbf{a}}{\to} a_1) + \ldots + \Pr(a_N)(a_N \overset{\mathbf{a}}{\to} a_1); \\
\tau_2(\mathbf{a}) &= \omega(a_2) + \omega(a_3)(a_3 \overset{\mathbf{a}}{\to} a_2) + \ldots + \omega(a_N)(a_N \overset{\mathbf{a}}{\to} a_2) \\
&= \Pr(a_2)(a_2 \overset{\ell=0}{\to} a_2) + \Pr(a_3)(a_3 \overset{\mathbf{a}}{\to} a_2) + \ldots + \Pr(a_N)(a_N \overset{\mathbf{a}}{\to} a_2).
\end{aligned}$$

sample centrality is defined as in Definition 4, with the caveat that (i) if (a_1, a_1) or (a_2, a_2) is drawn, only the zero-length walk of weight 1 is considered; (ii) if (a_1, a_2) or (a_2, a_1) is drawn,

only the path $\pi = (a_1, a_2)$ *(with* $\kappa(\pi) = \sigma(a_1, a_2)$*) or* $\pi = (a_2, a_1)$ *(with* $\kappa(\pi) = \sigma(a_2, a_1)$*) is considered respectively.*

H.3 Path-summability

If walk-summability fails for at least some $\mathbf{a} \in \mathcal{A}_k$, the sample precision matrix $\Sigma^{-1}(\mathbf{a})$ is no longer interpretable in terms of walks traversing the subgraph of the sample as argued in section H.1. This is problematic for the definition of sample centrality proposed in definition 4. The following example illustrates this for samples of size two.

Example H.2 (Failure of walk-summability for small samples)**.** *Any* $\mathcal{G}$ *is* 1*-walk-summable but not* 2*-walk-summable. Walk-summability for* $n = 1$ *is straightforward as for any* $\mathbf{a} = (a_1)$*, only a zero-length walk within* $\mathbf{a}$ *exists. For* $n = 2$*, the largest eigenvalue of* $\bar{A}(\mathbf{a})$ *for any* $\mathbf{a} = (a_1, a_2)$ *is* $1 + |\sigma(a_1, a_2)| \geq 1$*. Hence,* $\rho(\mathbf{a}) \geq 1$*. By lemma H.1 walk-summability fails.*

To circumvent this issue, we turn to an alternative interpretation of the precision matrix in terms of path sums following results developed in Giscard et al. (2016). The path-sum formulation is derived from a fundamental algebraic property of the set of all walks on a weighted graph: any walk in this graph factorizes uniquely into products of paths and cycles (as defined in section H.1).[8]

Lemma H.4 provides a recursive expression for each entry in $\Sigma^{-1}(\mathbf{a})$ in terms of finite alternating sums of paths and cycles. In the following construction, each node in $\mathcal{G}$ is endowed with an additional self-loop of weight $\sigma(a, a)$ normalized to unity. For a given $\mathbf{a}$, let $\mathcal{G}_{\mathbf{a}}$ be the subgraph restricted to it. Let $\mathcal{G}_{\mathbf{a}\setminus S}$ denote the subgraph of $\mathcal{G}_{\mathbf{a}}$ after deleting nodes in $S \subset \mathbf{a}$ and all adjacent edges. Also, $\Pi_{\mathcal{G}_{\mathbf{a}}, a_i a_j}$ and $\Gamma_{\mathcal{G}_{\mathbf{a}}, a_i a_j}$ denote respectively the set of all paths and cycles in $\mathcal{G}_{\mathbf{a}}$ from a_i to a_j for $i, j \leq k$. Let $\ell(\cdot)$ denote the length of a path/cycle. For any $\gamma \in \Gamma_{a_j a_j}$, $\gamma = (a_j, \gamma_2, \ldots, \gamma_{\ell(\gamma)}, a_j)$, and for $\pi \in \Pi_{a_i a_j}$ $\pi = (a_i, \pi_2, \ldots, \pi_{\ell(\pi)}, a_j)$.

Lemma H.4 (Path-sum formulation, Giscard et al. (2016))**.** *Fix sample* $\mathbf{a} = (a_1, \ldots, a_k)$ *and let* $\Sigma^{-1}(\mathbf{a})$ *be its positive definite precision matrix. The entries of the precision matrix* $\Sigma^{-1}(\mathbf{a})$ *are given by the recursive summation over paths and cycles:*

$$\Sigma^{-1}_{a_i a_j} = \sum_{\pi \in \Pi_{\mathcal{G}_{\mathbf{a}}, a_i a_j}} (-1)^{\ell(\pi)} \prod_{m=1}^{\ell(\pi)+1} \left\{ \left(\Sigma_{\mathcal{G}_{\mathbf{a}\setminus a_j, \pi_2, \ldots, \pi_{m-1}}} \right)^{-1}_{\pi_m \pi_m} \sigma(\pi_m, \pi_{m+1}) \right\} \Sigma^{-1}_{a_j a_j}, \tag{5}$$

$$\Sigma^{-1}_{a_j a_j} = \left(\sum_{\gamma \in \Gamma_{a_j a_j}} (-1)^{\ell(\gamma)+1} \sigma(a_j, \gamma_{\ell(\gamma)}) \prod_{m=2}^{\ell(\gamma)} \left\{ \left(\Sigma_{\mathcal{G}_{\mathbf{a}\setminus a_j, \gamma_2, \ldots, \gamma_{m-1}}} \right)^{-1}_{\gamma_m \gamma_m} \sigma(\gamma_m, \gamma_{m+1}) \right\} \right)^{-1}. \tag{6}$$

[8] This property holds for both walk-summable and non-walk summable graphs.

Proof of lemma H.4. The result directly follows from theorem 2 in Giscard et al. (2016), by replacing J with $\Sigma(\mathbf{a})$. □

Each summation in lemma H.4 has finitely many terms because there are only finitely many paths and cycles in a finite graph $\mathcal{G}_{\mathbf{a}}$. Note the recursive structure of this formulation: the diagonal terms of the precision matrix are obtained by repeatedly applying (6) and this recursion ends in finite time. Hence, any diagonal term of the precision matrix is expressed in terms of products of cycles in subgraphs of $\mathcal{G}_{\mathbf{a}}$. Equation (5) uses equation (6) to obtain the off-diagonal terms of the sample precision matrix: they combine paths and cycles in subgraphs of $\mathcal{G}_{\mathbf{a}}$. This path formulation holds both when $\mathcal{G}$ is $\mathbf{a}$-walk-summable and when it is not.

Example H.3. *Following up on the observation made in example H.2, let $k = 2$ and consider the sample subgraph in Figure 7. We compute the entries of the sample precision matrix using equations (5) and (6). First, note that there are two cycles from a_1 to itself: $\gamma = (a_1, a_1)$*

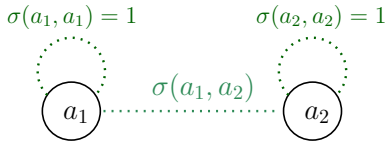


Figure 7: The subgraph for sample $\mathbf{a} = (a_1, a_2)$.

and $\gamma' = (a_1, a_2, a_1)$. Therefore,

$$p_{11} = \left(\underbrace{(-1)^2 \sigma(a_1, a_1)}_{\text{cycle } \gamma} + \underbrace{(-1)^3 \sigma(a_1, a_2) \left(\Sigma_{\mathcal{G}_{\mathbf{a} \setminus a_1}}\right)^{-1}_{22} \sigma(a_2, a_1)}_{\text{cycle } \gamma'} \right)^{-1}$$
$$= \left(\sigma(a_1, a_1) - \sigma(a_1, a_2)^2 \frac{1}{\sigma(a_2, a_2)} \right)^{-1} = \frac{\sigma(a_2, a_2)}{\sigma(a_1, a_1)\sigma(a_2, a_2) - \sigma(a_1, a_2)^2} = \frac{1}{1 - \sigma(a_1, a_2)^2}.$$

By similar reasoning, we obtain p_{22}, which is also a combination of two cycles. Next, there is only one path from a_1 to a_2: $\pi = (a_1, a_2)$. Hence,

$$p_{12} = (-1)^2 \underbrace{\left(\Sigma_{\mathcal{G}_{\mathbf{a} \setminus a_1}}\right)^{-1}_{22}}_{\text{self-loop } \gamma'' = (a_2, a_2)} \sigma(a_1, a_2) p_{22} = \frac{1}{\sigma(a_2, a_2)} \sigma(a_1, a_2) \frac{\sigma(a_2, a_2)}{\sigma(a_1, a_1)\sigma(a_2, a_2) - \sigma(a_1, a_2)^2}$$
$$= \frac{\sigma(a_1, a_2)}{1 - \sigma(a_1, a_2)^2}.$$

That is, the sign-alternating path product $(-1)^2 \sigma(a_1, a_2)$ is weighted by $p_{22}/\sigma(a_2, a_2)$.

Sample centrality redefined. The $(i,j)^{th}$ entry of the precision matrix for the sample is now expressed as the alternating sum of the path products of the form $(-1)^{\ell(\pi)}\kappa(\pi)$ for all paths π between $a_i \in \mathbf{a}$ and $a_j \in \mathbf{a}$, where each path is weighted as in lemma H.4. Therefore, $\tau_j(\mathbf{a})$ should be reinterpreted as the expected alternating path sum from a randomly drawn $a \in \mathcal{A}$ to $a_j \in \mathbf{a}$, where all nodes in the path from the second node are sample nodes. Correspondingly, centrality γ is reinterpreted as the *expected alternating path sum* from a randomly drawn attribute-node $a \in \mathcal{A}$ to another randomly drawn attribute $a' \in \mathcal{A}$.

H.4 Relation to other centrality measures

Sample centrality departs from most existing centrality measures insofar as it is defined over subsets of nodes rather than only single nodes. Notable exceptions are Everett and Borgatti (1999) (who introduce knotty centrality), Shanahan and Wildie (2012) (who extend degree / closeness / betweenness centrality to group counterparts), and Prummer (2019) (who extends weighted degree centrality to subsets of nodes in a model of media targeting). In particular, sample centrality is a striking amalgam of betweenness centrality and Katz-Bonacich centrality. We compare it to each in turn.

Relation to betweenness measures. Betweenness centrality captures the extent to which a certain node lies on the paths between any two other nodes. Although the original betweenness measure took into account only geodesic paths between any two nodes (Freeman (1977)), subsequent variations extended it to weighted graphs and non-geodesic paths (flow betweenness in Freeman, Borgatti and White (1991)), random walks from a fixed source to a fixed sink (random-walk betweenness in Newman (2005)), and betweenness of subsets of nodes (group betweenness in Everett and Borgatti (1999)). In the spirit of betweenness measures, sample centrality captures how strongly a sample of nodes connects any two nodes in the graph – yet with some crucial differences.

First, for $k = 1$, that is, for a singleton sample $\mathbf{a} = (a_1)$, sample centrality looks at the shortest path of the form $a \to a_1 \to a'$, but weighs this path by $\omega(a)\omega(a')$ rather than by the inverse of the total number of geodesic paths from a to a' as betweenness centrality would. This weighting of the pair (a, a') generalizes to $k \geq 1$ as well.

Second, for $k \geq 1$ sample centrality considers inference channels of a similar form $a \to$ sample $\to a'$, but these are walks rather than paths. The walks are constrained to travel only within the sample before ending at a'. This stands in contrast to the construction in Everett and Borgatti (1999) where group betweenness counts only the share of geodesic paths from a to a' that go through sample nodes.

Relation to Bonacich centrality. Sample centrality features alternating walk sums of any length between any two sample nodes, because the covariance of the sample is $\Sigma^{-1}(\mathbf{a}) = (I + A(\mathbf{a}))^{-1}$. In the case in which the graph is $\mathbf{a}$-walk-summable, this is reminiscent of Katz-Bonacich centrality, which counts the sum of walks of any length emanating from a given

node (Katz (1953), Bonacich (1987)). Katz-Bonacich centrality discounts walks of length ℓ by β^ℓ, where $\beta \in (0,1)$ is the discount factor. In contrast, sample centrality discounts walks of length ℓ by $(-1)^\ell$.[9]

More importantly, the weight $\tau_j(a;\mathbf{a})$ that scales sample observation $f(a_j)$ in the expectation of $f(a)$ (equation (4)) can be interpreted in Katz-Bonacich-like terms. The Katz-Bonacich centrality of node $a_j \in \mathbf{a}$ within the sample subgraph $\mathcal{G}_\mathbf{a} = (\mathbf{a}, E_\mathbf{a})$ is

$$\sum_{a_i \in \mathbf{a}} (a_i \overset{\mathbf{a}}{\to} a_j),$$

whereas for a given out-of-sample node a,

$$\tau_j(a;\mathbf{a}) = \sum_{a_i \in \mathbf{a}} \sigma(a, a_i)(a_i \overset{\mathbf{a}}{\to} a_j).$$

So $\tau_j(a;\mathbf{a})$ captures a modified notion of Katz-Bonacich centrality of sample node a_j: walks from a_j to other sample nodes $a_i \in \mathbf{a}$ are weighted by $\sigma(a, a_i)$ rather than equally. If $\sigma(a, a_i)$ is the same for any $a_i \in \mathbf{a}$, $\tau_j(a;\mathbf{a})$ simplifies to Katz-Bonacich centrality. Therefore, $\tau_j(\mathbf{a})$ is the modified Katz-Bonacich centrality of node $a_j \in \mathbf{a}$ averaged out across all possible starting nodes $a \in \mathcal{A}$.

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[9] The alternating sign captures the opposing effects of mediating and moderating variables, the net effect of which is reflected in the total covariance between the first node and the last node in a path or walk.

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